S4–01

Gain–Phase Type Robust Control Theory

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Summary

In this paper, we have studied the GP–robust control theory that uses both of the gain and phase uncertainties. We've made some contributions to proving the GP–robust stability theorem that overcomes the conservatism of the conventional small gain theorem ignoring phase uncertainty, formulating the GP–mixed sensitivity problem that improves robust performance and suggesting the W_Ω–K iterative algorithm as its solving method. Then, we have newly formulated Ψ–control and H^∞/Ψ–mixed control problems using gain and phase design indices. Finally we have illustrated the effectiveness of the proposed method through design examples.

Keywords: Gain–phase type robust control theory, GP–robust stability theorem, GP–mixed sensitivity problem, W_Ω–K iterative algorithm, Ψ–control, H^∞/Ψ–mixed control

1. Introduction

Though the postmodern control theory with H^∞–control as core has achieved the great success theoretically and practically, it has a series of limitations, especially essential drawbacks of ignoring phase information. The postmodern control uses only the gain information and ignores the phase information by using the H^∞–norm as optimality criterion and small–gain theorem as stability condition.[3, 12] Therefore H^∞–robust control theory has essential limitations in the achievable control performance.

Considering the history on the research of phase information since 1980, the paper [10] which was proposed simultaneously with the first paper on H^∞–control [3] emphasized the importance of phase together with gain, and opened the beginning of the study on phase of multivariable systems based on results of [1] for singular value of matrix. However, it did not draw the considerable attention because the phase was difficult to deal mathematically and the 1980s was the golden times for H^∞–control and μ–theory.

Study on the phase of multivariable systems began to be deepened gradually ushering in 1990s after H^∞–control theory had been completed in 1989. The phase margin of MIMO system was studied in [2], and the phase μ–theory considering the phase uncertainty was proposed in [11]. On the other hand, the theory for the robust stability and the robust performance analysis named "Neo Robust Control" was studied in earnest by K. Z. Liu and his colleagues in these days and the guide to loop shaping was proposed. But their theories were limited to SISO systems and the design theory was not suggested.[5~8]

In this paper, we newly propose Gain–phase type robust control theory which uses both of the gain and phase information simultaneously and formulate Ψ–control and H^∞/Ψ–mixed control problems using gain and phase design indices ([9]).

2. GP–robust stability condition and GP–mixed sensitivity method

2.1 Preliminaries

Let us consider the feedback system depicted in Fig. 1.

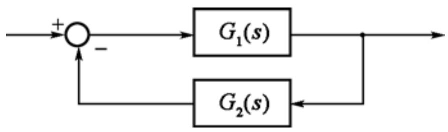

Figure 1. Block diagram of feedback system

The unit feedback system whose open–loop transfer function is $G(s)$ is stable if and only if

$$\sum_{k=1}^{n} n_k = 0 \tag{1}$$

where, n_k is the number that the eigenvalues $g_k(j\omega)$ of $G(j\omega)$ enclose the critical point $(-1, j_0)$ in according to $\omega \in (-\infty, \infty)$.[10]

Using the above relation, we can obtain the following theorem. [9]

Theorem 1. When $G_1(s)$ and $G_2(s)$ are the stable transfer functions, the feedback system is stable if

$$\overline{\sigma}[G_1(j\omega)] \cdot \overline{\sigma}[G_2(i\omega)] < 1, \ \forall \omega \in \Omega \tag{2}$$

Where

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \tag{3}$$

$$\begin{aligned} \Omega_1 = \{\omega \,|\, &[\underline{\varphi}(G_1(j\omega)) + \underline{\varphi}(G_2(j\omega)) - \psi_m(\omega) \le \pi] \wedge \\ &[\overline{\varphi}(G_1(j\omega)) + \overline{\varphi}(G_2(j\omega)) + \psi_m(\omega) \ge \pi] \wedge \\ &[\Delta\varphi(G_1(j\omega)) + \Delta\varphi(G_2(j\omega)) < \pi], \ \omega \in R_+\} \\ \Omega_2 = \{\omega \,|\, &[\Delta\varphi(G_1(j\omega)) + \Delta\varphi(G_2(j\omega)) \ge \pi], \ \omega \in R_+\} \\ \Omega_3 = \{\omega \,|\, &1 \le [c_1(j\omega) - 1] \cdot c_2(j\omega), \ \omega \in R_+\} \end{aligned}$$

$$c_1(j\omega) = cond[G_1(j\omega)], \ c_2(j\omega) = cond[G_2(j\omega)]$$

and the symbols $\cup$ notate the union operation and $\wedge$ the logical product, $\Delta\varphi$ is the phase spread, and $R_+ = [0, \infty)$.

The maximum value of phase modification is $\psi_m = \tan^{-1}\{([c_1-1]\cdot c_2)/(1-[c_1-1]\cdot c_2)\}$ under the assumption $1 > [c_1-1]\cdot c_2$.

Theorem 1 was proved in [9], thus the proof is omitted here.

2.2 GP–robust stability condition

Let the transfer function of the plant be described as $G(s)=[I+\Delta G(s)]\cdot G_0(s)$ and the polar decomposition of $\Delta G(s)$ as $\Delta G_s=\Delta H(s)\cdot\Delta U(s)=\Delta H(s)\cdot e^{j\Delta F(s)}$. Then it is supposed that the gain characteristics of $\Delta G(s)$ satisfies $\sigma[\Delta G(j\omega)]=\lambda_{\max}[\Delta_H(j\omega)]\leq|W_T(j\omega)|$, where $W_T(s)$ is the SISO minimum phase transfer function. If we introduce $\Delta(s)=\Delta G(s)/W_T(s)=[\Delta H(s)/W_T(s)]\cdot e^{j\Delta F(s)}$, we can obtain $G(s)=[I+W_T(s)\Delta(s)]\cdot G_0(s)$.

After all, the uncertainty of $\Delta(s)$ consists of the normalized gain uncertainty $\Delta H(s)/W_T(s)$ and the phase uncertainty $\Delta F(s)$.
Gain uncertainty bound:

$$\overline{\sigma}[\Delta(j\omega)]<1 \tag{4}$$

Phase uncertainty bound:

$$\left.\begin{array}{l}\overline{\varphi}[\Delta G(j\omega)\leq\overline{q}(\omega)\\ \underline{\varphi}[\Delta G(j\omega)]\geq q(\omega)\end{array}\right\} \tag{5}$$

where $\overline{q}(\omega)$ and $q(\omega)$are single variable functions satisfying $\overline{q}(\omega),\ q(\omega):R\rightarrow[0,\ 2\pi)$.

Let us find the robust stability condition of the feedback control system in Fig. 2.

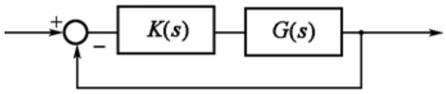

Figure 2. Block diagram of the feedback control system

The plant $G(s)$ is assumed to have the multiplicative uncertainty and $\Delta G(s)$ is assumed to belong to the following perturbation set Ξ.

$$\Xi=\Xi_S\cup\Xi_U \tag{6}$$

$\Xi_S=\{\Delta G(s)\in RH_\infty|$ satisfies Eq. (4) and (5)$\}$
$\Xi_U=\{\Delta G(s)\in RL_\infty|N_+(G)=N_+(G_0)$ satisfies Eq. (4) and (5)$\}$,
$N_+(G)$– the number of unstable poles of $G(s)$

We can prove the following theorem easily with the well–known internal stability condition [12] of the feedback control system.

Theorem 2 (GP–robust stability condition)

The feedback control system composed of $G(s)$ and nominal stabilizing controller $K(s)$ with arbitrary $\Delta G(s)\in\Xi$ is stable if

$$\overline{\sigma}[W_T(j\omega)T(j\omega)]<1,\ \forall\omega\in\Omega \tag{7}$$

where

$$\Omega=\Omega_1\cup\Omega_2\cup\Omega_3 \tag{8}$$

$$\begin{aligned}\Omega_1=\{\omega\,|\,&[\varphi(T(j\omega))+\underline{q}(\omega)-\hat{\psi}_m(\omega)\leq\pi]\wedge\\ &[\overline{\varphi}(T(j\omega))+\overline{q}(\omega))+\hat{\psi}_m(\omega)\geq\pi]\wedge\\ &[\Delta\varphi(T(j\omega))+\overline{q}(\omega)-\underline{q}(\omega)<\pi],\ \omega\in R_+\}\end{aligned}$$

$$\Omega_2=\{\omega\,|\,[\Delta\varphi(T(j\omega))+q(\omega)-\underline{q}(\omega)\geq\pi],\ \omega\in R_+\}$$

$$\Omega_3=\{\omega\,|\,1\leq[c_1(\omega)-1]\cdot c_2(j\omega),\ \omega\in R_+\}$$

where $T(s)=G_0(s)K(s)[I+G_0(s)K(s)]^{-1}$, c_1 and c_2 are the condition numbers of $T(s)$ and $\Delta G(s)$ respectively.

The proof of Theorem 2 was proposed in [9].

Comparing with the small–gain condition, the GP–robust stability condition is much relaxed by decreasing the frequency range from the real number range R to the frequency set Ω.

2.3 GP–mixed sensitivity problem

The robust performance condition is generally formulated as follows [12].

Performance condition:

$$\|W_S(s)\cdot S(s)\|_\infty\rightarrow\min \tag{9}$$

Robust stability condition:

$$\|W_T(s)\cdot S(s)\|_\infty<1 \tag{10}$$

Therefore, the robust performance problem can be transformed approximately into the mixed sensitivity problem

$$\left\|\begin{matrix}\rho W_S(s)\cdot S(s)\\ W_T(s)\cdot T(s)\end{matrix}\right\|<1 \tag{11}$$

But the mixed sensitivity method is based on the small–gain robust stability condition (10), thus it is conservative and has the limitation in improving performance, therefore there exists the possibility to improve performance using condition (7).

Let us make some following operations to solve the above problem.

First, define the following characteristic function $\chi_\Omega(j\omega)$ of the frequency set Ω of GP–robust stability theorem as

$$\chi_\Omega(\omega)=\begin{cases}1,&\omega\in\Omega\\0,&\omega\notin\Omega\end{cases} \tag{12}$$

Then introduce the SISO minimum phase transfer function $d_\Omega(s)$ satisfying the following condition.

$$|\,d\Omega(j\omega)|\approx\chi_\Omega(\omega) \tag{13}$$

$W_\Omega(s)$ is calculated by using $d_\Omega(s)$ as

$$W_\Omega(s)=d_\Omega(s)W_T(s) \tag{14}$$

After all, $W_\Omega(s)$ is a gain–phase uncertainty weighting function composed of $W_T(s)$ for the gain uncertainty and $d_\Omega(s)$ for the phase uncertainty.

Then, the GP–robust stability condition can be written approximately as

$$\|W_\Omega(s)\cdot T(s)\|_\infty<1 \tag{15}$$

Using Eq. (9) and (15), the following GP–mixed sensitivity problem is formulated as

$$\left\|\begin{matrix}\rho W_S(s)\cdot S(s)\\ W_\Omega(s)\cdot T(s)\end{matrix}\right\|_\infty<1 \tag{16}$$

In order to solve the GP–mixed sensitivity problem, the following algorithm was proposed [9].

W_{Ω}–K iteration algorithm

(i) Let i=0. Calculate the controller $K(s)$ satisfying the small–gain mixed sensitivity problem (11) by γ–iteration method of H^{∞}–control theory. Then, let the obtained controller be $K_i(s)$, and the value of ρ be ρ_i.

(ii) Decide the frequency set Ω_i, based on the controller $K_i(s)$.

(iii) Defining the characteristic function of Ω_i and deciding the SISO transfer function $d_{\Omega}(s) \in RH_{\infty}$ satisfying Eq. (13), then calculate $W_{\Omega}(s) = d_{\Omega}(s) W_T(s)$.

(iv) Design the controller $K_{i+1}(s)$ satisfying the GP–mixed sensitivity condition (16) by H^{∞}–design method. Here ρ is taken in the range in which there exists the controller satisfying Eq. (16) and the relationship $\rho \geq \rho_i$ is satisfied. Let the obtained ρ be ρ_{i+1}.

(v) Decide the frequency set Ω_{i+1} based on the controller $K_{i+1}(s)$.

(vi) If $\Omega_{i+1} \subset \Omega_i$, $\rho_{i+1} > \rho_i$ then $\Omega_{i+1} \to \Omega_i$, $K_{i+1} \to K$, $\rho_{i+1} \to \rho_i$ and go to step (iii), and if not, go to step (vii).

(vii) Let $K(s)=K_i(s)$, $\rho=\rho_i$ and conclude the iteration.

3. Ψ–Control and H∞/Ψ–Mixed Control Problem

3.1 Formulation of Ψ–Control Problem

In the view point of H^{∞}–control and several control strategies based on it, only the information of gain matrix $H(s)$ is used in the polar decomposition description of the transfer function $G(s)$, $G(s)=H(s)\cdot e^{jF(s)}$.

Furthermore, there does not exist as 1: 1 relation between $H(s)$ and $F(s)$, which is the essential drawback of H^{∞}–control theory. To overcome this conservatism, we propose Ψ–control problem corresponding to H^{∞}–control problem in this paper.

Let us consider the standard block diagram depicted in Fig. 3.

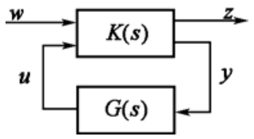

Figure 3. Standard block diagram of Ψ–control problem

As in H^{∞}–control problem, w denotes the exogenous signal, z the controlled variable, u the control variable and y the measured variable. And P is a generalized plant and satisfies the following input–output relation.

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \quad (17)$$

In Fig. 3, the controller $K(s)$ satisfies $u(s) = K(s) \cdot y(s)$ and the transfer function of the feedback system is written as

$$z(s) = \Phi(s) \cdot w(s) \quad (18)$$

$$\begin{aligned} \Phi(s) &= p_{11} + p_{12} K (I - p_{22} K)^{-1} p_{21} = \\ &= p_{11} + p_{12} (I - K p_{22}{}^{-1}) K p_{21} \end{aligned} \quad (19)$$

Then Ψ–control problem is formulated as follows.

Definition 1 (Ψ–control problem)

In the feedback system depicted in Fig. 3, discriminate whether there exist the controllers satisfying the following conditions, if exist, Ψ–control problem is defined as to find the whole such controllers.

(i) to ensure the internal stability of the feedback system

(ii) to meet phase condition:

$$\Psi(\Phi) < \beta \quad (20)$$

In Eq. (20), $\Psi(\Phi)$ is the phase of the transfer function $\Phi(s)$ and it is defined as follows [9].

$$\Psi(\Phi) = \sup_{\omega \in R} \psi_{\Phi}(j\omega) = \sup_{\omega \in R} \max_i \{ | \lambda_i [F(j\omega)] | \} \quad (21)$$

3.2 H∞/Ψ–mixed control problem

The most ideal method for control system design is to consider both the gain and phase simultaneously as design indices. To do that, let us consider the block diagram depicted in Fig. 4. In Fig. 4, z_1 and z_2 are controlled variables for H^{∞}–control and Ψ–control respectively.

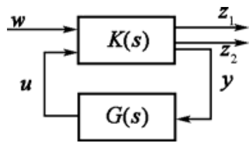

Figure 4. Block diagram of H∞/Ψ–mixed control problem

Then, H∞/Ψ–mixed control problem is formulated as follows.

Definition 2 (H∞/Ψ–mixed control problem)

In the feedback system depicted in Fig. 4, H∞/Ψ–control problem is defined as to find the whole controllers $K(s)$ satisfying the following conditions.

(i) to ensure the internal stability of the feedback system

(ii) to meet gain condition:

$$\| \Phi_{z1w} \|_{\infty} < \gamma \quad (22)$$

and phase condition:

$$\psi(\Phi_{z2w}) < \beta \quad (23)$$

4. Design example of the fuel injection pump robust control system

In order to examine the effectiveness of the proposed method, using the GP–mixed sensitivity method, we redesign the fuel injection pump control system which has been previously designed by the conventional mixed sensitivity method in [4], and compare two control systems. In [4], the transfer functions in the operation temperature 0°C, 25°C and 60°C are obtained by the instrumental variable method as follows.

$$G_{25}(s) = \frac{5.498 s^2 + 400.7 s - 444\ 400}{s^3 + 93.73 s^2 = 9\ 520 s + 121\ 400}$$

$$G_{00}(s) = \frac{-0.0173\ 6s^2 + 493.9s - 313\ 700}{s^3 + 98.34s^2 + 9\ 223s + 87\ 710}$$

$$G_{60}(s) = \frac{4.677s^2 - 285.9s - 505\ 300}{s^2 + 91.53s^2 + 10\ 080s + 176\ 200}$$

If $G_{25}(s)$ is taken as a nominal model and the model variation with temperature change can be described as the multiplicative uncertainty model, i.e., $G(s)=G_0(s)[1+\Delta G(s)]$, $\Delta G(s)=[G(s)-G_0(s)]$.

The weighting functions are selected as $W_S(s)=1/s$, $W_T(s)=s/100$. Then, we design the controller by using hinfsyn.m of MATLAB by increasing the value of ρ from 1.

We use the ε–avoidance method to meet the assumptions of the standard H^∞–control problem and change $W_S(s)$ into $W_S(s)=1/(s+\varepsilon)$, ε=0.01. Then the maximum value of ρ is 56.8 and the H^∞–controller is obtained as follows.

$$K_1(s) = \frac{-1.435\cdot 10^8 s^3 - 1.345\cdot 10^{10} s^2 - 1.367\cdot 10^{12} s - 1.743\cdot 10^{13}}{s^4 + 7.893\cdot 10^6 s^3 + 6.026\cdot 10^9 s^2 + 1.123\cdot 10^{12} s + 1.123\cdot 10^9}$$

Designing the controller by using W_Ω–K iteration, the 9th order controller is obtained and the value of ρ reaches 105.1.

$$K_2(s) = \frac{\begin{gathered}-3.468\cdot 10^6 s^8 - 2.135\cdot 10^{10} s^7 - 2.53\cdot 10^{12} s^6 - \\ -2.739\cdot 10^{14} s^5 - 9.813\cdot 10^{15} s^4 - 2.922\cdot 10^{17} s^3 - \\ -2.924\cdot 10^{18} s^2 - 2.227\cdot 10^{17} s - 5.078\cdot 10^{17}\end{gathered}}{\begin{gathered}s^9 + 1.916\cdot 10^5 s^8 + 4.489\cdot 10^8 s^7 + 4.059\cdot 10^{11} s^6 + \\ +9.791\cdot 10^{13} s^5 + 2.628\cdot 10^{15} s^4 + 1.014\cdot 10^{17} s^3 + \\ +6.543\cdot 10^{15} s^2 + 1.77\cdot 10^{16} s + 1.769\cdot 10^{13}\end{gathered}}$$

In order to evaluate the effectiveness of the proposed method in the time domain, we compare the step response of $T_1(s)$ and $T_2(s)$ (Fig. 5).

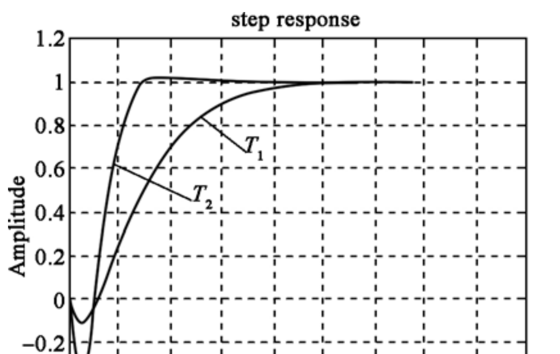

Figure 5. Step response of $T_1(s)$ and $T_2(s)$

5. Conclusions

In this paper we have resolved the following problems.

First, we have proved the GP–robust stability theorem which uses both of the gain and phase information. This is a development of the result of [10] and is more general than that of [3]. Also, it has an advantage that it deals with MIMO systems instead of SISO systems in [5~8].

Second, we have newly formulated the GP–mixed sensitivity method which can overcome the limitation of the small–gain one and proposed W_Ω–K iteration algorithm to solve it.

Third, we have newly formulated Ψ–control problem using phase design index and H^∞/Ψ mixed control problem using both of the gain and phase design indices. It remains our future work to solve Ψ–control and H^∞/Ψ mixed control problems.

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