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Relation between Shapley Value and Core in Games on Concept Lattices

*O Un Suk**

Faculty of Applied Mathematics, Kim Chaek University of Technology, Pyongyang, DPRK

*Corresponding author: Email: ous8511@star–co. net. kp

Summary

Game on concept lattices is the cooperative TU–game, where a concept is a pair (S, S') with S being a subset of players or objects, and S' a subset of attributes. Any such game induces a game on the set of players, which makes it to be a TU–game whose collection of feasible coalition is a lattice closed under intersection. In this paper, we introduce the concept of strong convexity in such induced games and prove that under such the assumption the extent Shapley value as proposed by Faigle and Grabisch (2016) is always in its extent core.

Keywords: game on concept lattice; Shapley value; core; strong convexity

1. Introduction

Cooperative games with transferable utility (TU–games) have been widely studied, and used in many domains of applications. N being a set of players, or more generally, a set of objects, a TU–game v: $2^N \to R$ with $v(\phi)=0$ assigns every coalition or group $S\subseteq N$ a number representing its "worth".

Shapley value and core are solution concepts well–known in cooperative game theory. The Shapley value yields a single distribution vector, satisfying a set of four natural axioms (pareto optimality, symmetry, linearity, null player property), while the core is a set of distribution vectors that are pareto optimality and satisfy coalition rationality (i. e., a coalition receives at least its own worth).

In many situations, however, not all subsets of N can be realized as coalitions or are feasible, which means that the mapping v is defined on a subcollection F of 2^N only. Pioneering works considering this situation are due to [1, 2, 4, 5].

Faigle and Grabisch [3] has introduced the cooperative TU–game, where a concept is a pair (S, S') with S being a subset of players or objects, and S' a subset of attributes, have proposed Shapley value with its axiomatization, and have studied the properties of core.

In this paper, we study the relation between Shapley value and core in games on concept lattices.

2. Games on concept lattice

We begin by recalling that a lattice is a partially ordered set (poset)(L, ≤), where ≤ is reflexive, antisymmetric and transitive, such that for any two elements x, $y\in L$, a supremum $x\vee y$ and infimum $x\wedge y$ exist. The dual partial order $\le^{\partial}$ is defined by $x\le^{\partial}y$ if and only if $y\le x$. The dual of the lattice is the poset (L, $\le^{\partial}$).

A context is a triple C=(N, M, I), where N is a finite nonempty set of objects, M is a finite set of attributes, and I: $N\times M\to\{0, 1\}$ is a binary relation defined by $I(I, a)=1$ if object $i\in N$ satisfies attribute $a\in M$, and 0 otherwise.

Let $C=(N, M, I)$ be a context. The intent of a subset of objects $S\subseteq N$ is defined as the set of attributes satisfied objects in S:

$$S'_C = \{a\in M \mid I(i, a)=1,\ \forall i\in S\}$$

Dually, the extent of any set of attributes $A\subseteq N$ is defined as the set of objects satisfying all attributes in A:

$$A'_C = \{i\in N \mid I(i, a)=1,\ \forall a\in A\}$$

A concept in C is a pair (S, A) with $S\subseteq N$ and $A\subseteq N$ such that $S=A'$ and $A=S'$.

We denote by L_C the set of all concepts in C, and endow it with a partial order ≤ defined by

$$(S,\ A)\le(T,\ B)\quad \textit{if}\ \ S\subseteq T$$

Given a context C and its concept lattice L_C, the lattice of extents $(L_C^N, \subseteq)$ is defined by the set

$$L_C^N = \{S\subseteq N \mid (S,\ S')\in L_C\}$$

Similarly, the lattice of intents $(L_C^M, \subseteq)$ is defined as the set

$$L_C^M = \{A\subseteq M \mid (A,\ A')\in L_C\}$$

Clearly, the lattices L_C, L_C^N, $(L_C^M)^{\partial}$ are isomorphic.

To each concept $(A,\ A')\in L_C$, we assign a number $v(A,\ A')\in R$ (its meaning could be benefit, cost, evaluation, certainty degree of occurrence, etc.). We call the pair (C, v) a cooperative game on concepts or concept game for short, and impose the restriction $v(\phi, M)=0$ whenever $(\phi, M)\in L_C$.

We derive from v two mappings $v_N: L_C^N\to R$ and $v_M: L_C^M\to R$ defined by

$$v_N(S)=v(S,\ S')-v(M',\ M)\qquad (S\in L_C^N)$$

$$v_M(A)=v(N,\ \phi)-v(A,\ A')\qquad (A\in L_C^M)$$

In this paper, we consider the lattice of extents L_C^N of a context C and the game v_n defined on it only.

In the future, we simply express v_n as v.

(Definition 1) [3] The extent Shapley value of (C, v), denoted by $\Phi(C, v)$, is defined to be the Shapley value of the game on extents, given by

$$\Phi_i(v)=\begin{cases}\dfrac{1}{ch(C)}\sum_{C\in CH(C)}\dfrac{v(P_C^i)-v(Q_C^i)}{\left|P_C^i\setminus Q_C^i\right|}, & i\notin M'\\[2ex] \dfrac{v(M')}{|M'|}, & i\in M'\end{cases}$$

where

$CH(C)$ –set of all maximal chains from the bottom M' to the top N

$ch(C)$ –cardinality of set $CH(C)$

P_C^i –the first set in the sequence containing i

Q_C^i –the last set in the sequence which does not contain i

Theorem1 [3] the extent Shapley value satisfies separable payoff axiom, efficiency axiom, macro player axiom, concatenation axiom and decomposition axiom.

Definition 2 [3] given a game (C, v), the following set is called the extent core.

$$core(C,\ v) = \{X \in R^N \mid X(S) \geq v(S),\ \forall S \in L_C^N \text{ and } X(N) = v(N)\}$$

3. Relation between the extent Shapley value and extent core

Definition 3. (C, v) is called a strongly convex game if for every $S, R \in L_C^N$, not being in the same chain,

$$\frac{v(S \vee R) - v(R)}{|(S \vee R) \setminus R|} \geq \frac{v(S) - v(S \wedge R)}{|S \setminus (S \wedge R)|} \tag{2}$$

Theorem 2. If the extent game (C, v)is a strongly convex game then the extent Shapley value $\Phi(C, v)$ is in its extent core.

Proof Efficiency

From theorem 1, for the top element N it holds true that $\Sigma_{i\in N}\Phi_i(C, v) = v(N)$

·Coalitional Rationality

We should prove that any element (set) S, not the top element in L_C^N, satisfies coalitional rationality.

First, let us prove bottom M' satisfies coalitional rationality.

In case $M'=\Phi$, it is obvious.

If $M'\neq\Phi$, then by (1) it follows

$$\sum\nolimits_{i\in M'} \Phi_i(v) = \sum\nolimits_{i\in M'} \frac{v(M')}{M'} = v(M')$$

Next, let us prove that any set S satisfies the coalitional rationality in the case that the elements covered by S satisfy it.

Let us consider the case that S is a join–irreducible element of T. Then, we divide the chain set $CH(C)$ into the set of the chains $CH(C')$ containing S and T, and the set of the other chains $CH(C'')$.

Let us consider R the last element in the sequence which does not contain the elements of $S\setminus T$ for any chain $C\in CH(C'')$. For S and R their supermum and infimum surely exist in L_C^N from the property of lattice. Also, $(S\vee R)\setminus R$ contains the elements of $S\setminus T$ because this lattice is closed under intersection.

Therefore, if we set $T'=S\wedge R$, then T' is an element (set) of the chain $C\in CH(C')$ too, and does not become the later element (set) of T. Thus, the set T' is satisfied coalitional rationality, too.

On the other hand, it holds true the following expression from the condition of strong convexity.

$$\frac{v(S \vee R) - v(R)}{|(S \vee R) \setminus R|} \geq \frac{v(S) - v(S \wedge R)}{|S \setminus (S \wedge R)|} = \frac{v(S) - v(T')}{|S \setminus T'|} \tag{3}$$

By the expression (3), we get

$$\begin{aligned}
&\sum\nolimits_{i\in S} \Phi_i(v) = \sum\nolimits_{i\in T'} \Phi_i(v) + \sum\nolimits_{i\in S\setminus T'} \Phi_i(v) \geq v(T') + \sum\nolimits_{i\in S\setminus T'} \Phi_i(v) = \\
&= v(T') + \frac{1}{ch(C)} \sum\nolimits_{i\in S\setminus T'} \sum\nolimits_{C\in CH(C)} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|} = \\
&= v(T') + \frac{1}{ch(C)} \{\sum\nolimits_{i\in S\setminus T'} [\sum\nolimits_{C\in CH(C')} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|} \\
&\quad + \sum\nolimits_{C\in CH(C'')} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|}]\} = \\
&= v(T') + \frac{1}{ch(C)} \{\sum\nolimits_{C\in CH(C')} \sum\nolimits_{i\in S\setminus T'} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|} \\
&\quad + \sum\nolimits_{C\in CH(C'')} \sum\nolimits_{i\in S\setminus T'} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|}\} = \\
&= v(T') + \frac{1}{ch(C)} \{\sum\nolimits_{C\in CH(C')} [v(S) - v(T')] \\
&\quad + \sum\nolimits_{C\in CH(C'')} \sum\nolimits_{i\in S\setminus T'} \frac{v(P_C^i) - v(Q_C^i)}{|P_C^i \setminus Q_C^i|}\} \geq \\
&\geq v(T') + \frac{1}{ch(C)} \{\sum\nolimits_{C\in CH(C')} [v(S) - v(T')] \\
&\quad + \sum\nolimits_{C\in CH(C'')} [v(S) - v(T')]\} = \\
&= v(T') + \frac{1}{ch(C)} \sum\nolimits_{C\in CH(C)} [v(S) - v(T')] = \\
&= v(T') + [v(S) - v(T')] = v(S).
\end{aligned}$$

Let us consider the case that S is not a join–irreducible element of T. Then, let us consider $T_1, T_2 \ldots T_k$ the elements covered by S. We divide the chain set $CH(C)$ into the set of the chains $CH(C')$ containing S and $T_j,\ j = \overline{1, k}$, and the set of the other chains $CH(C'')$.

Similar to the above, if we choose the set R then we can prove that $\sum_{i\in S} \Phi_i(v) \geq v(S)$.

Conclusion

In this paper, we have introduced the concept of strong convexity, and have proved that the extent Shapley value is contained by its extent core under the condition of strong convexity in multichoice games on concept lattices.

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References

1. Faigle U. et al, 1992, The shapley value for cooperative games under precedence constraints, *Int. J. of Game Theory* 21, 249–266.
2. Faigle U., 1989, Cores of games with restricted cooperation, ZOR, *Methods Models Oper*.33 451–459.
3. Faigle U. et al, 2016, Games on concept lattices: Shapley value and core, *Discrete Applied Mathematics* 198, 29–47.
4. Grabisch M., 2013, The core of games on ordered structures and graphs, *Annal of Operation Research* 204 33–64.
5. Grabisch M. et al, 2007, A new approach to the core and Weber Set of multichoice games, *Mathematical Methods of Operation Research* 66 491–512.
6. Monjardet B., 2003, The presence of lattice theory in discrete problems of mathematical social sciences, Why, Mathematical Social 43, *Sciences* 46, 103–144.